An Introduction to Market Design

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CS 396 : Computational economics

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  - To some population,
  - When cannot satisfy everyone.

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- Classical answer :
  - Markets are not that bad.

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Figure: lesoir.be

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    - Problem : no resale = loose efficiency, resale = speculation.

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- Early literature : DA is preferable to BOS

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Theorem (Kesten (2010))

No Pareto efficient and strategy-proof mechanism Pareto dominates DA

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- Lab experiments (including large scale experiments).
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- Problem with living donor donation : organ compatibility.

Two-way kidney exchange

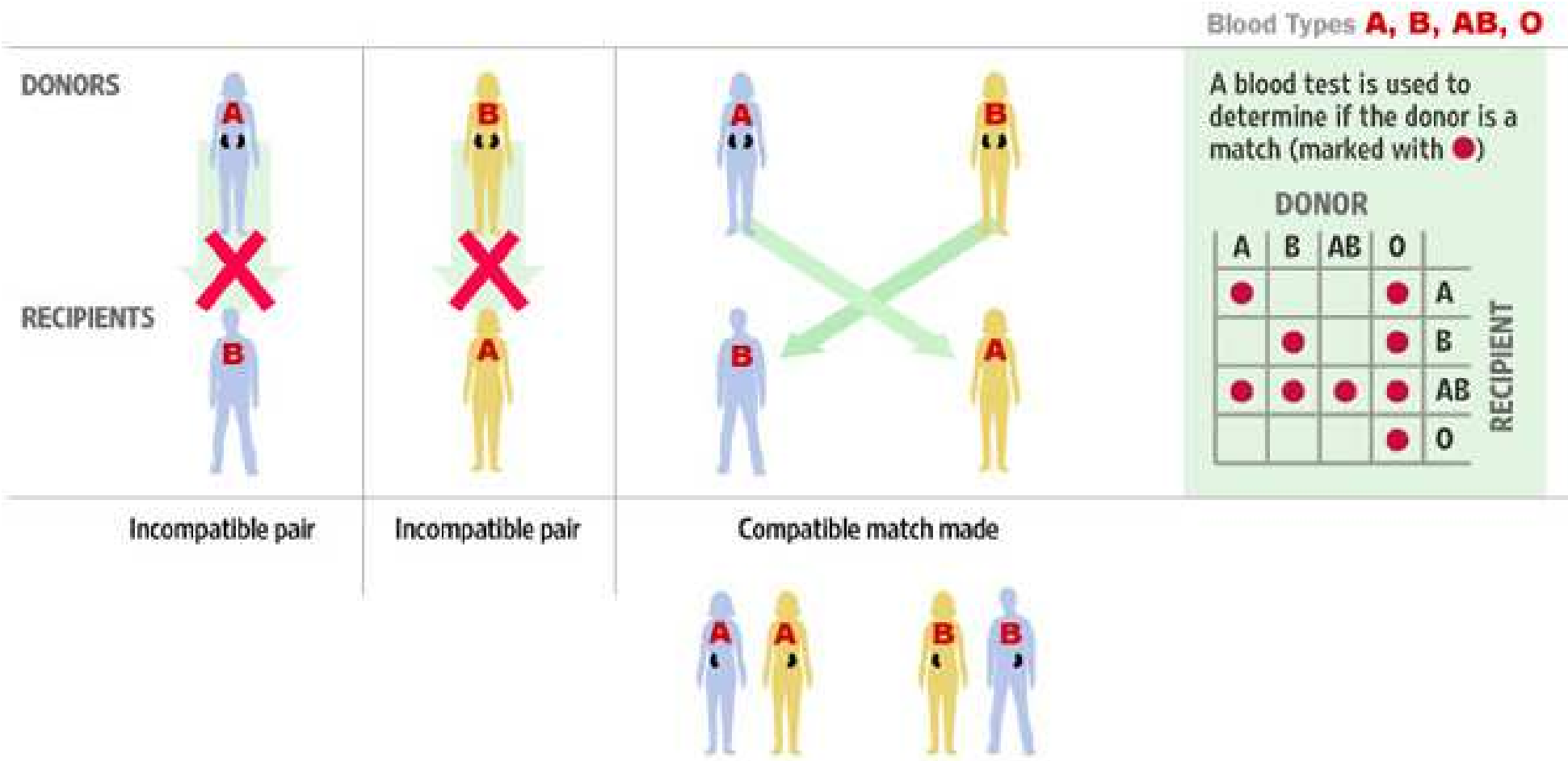
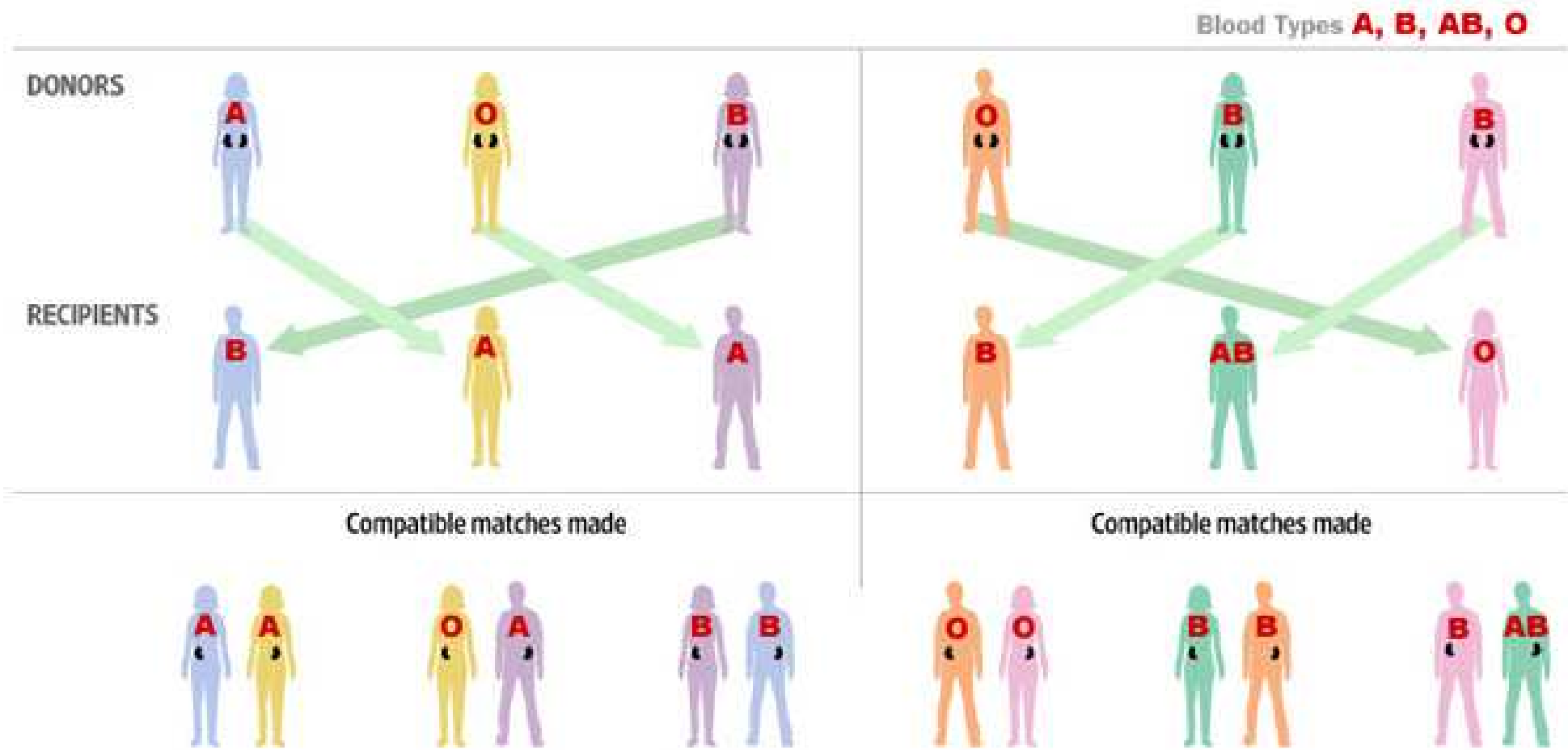


Figure: www.matchingdonors.com

Three-way kidney exchanges



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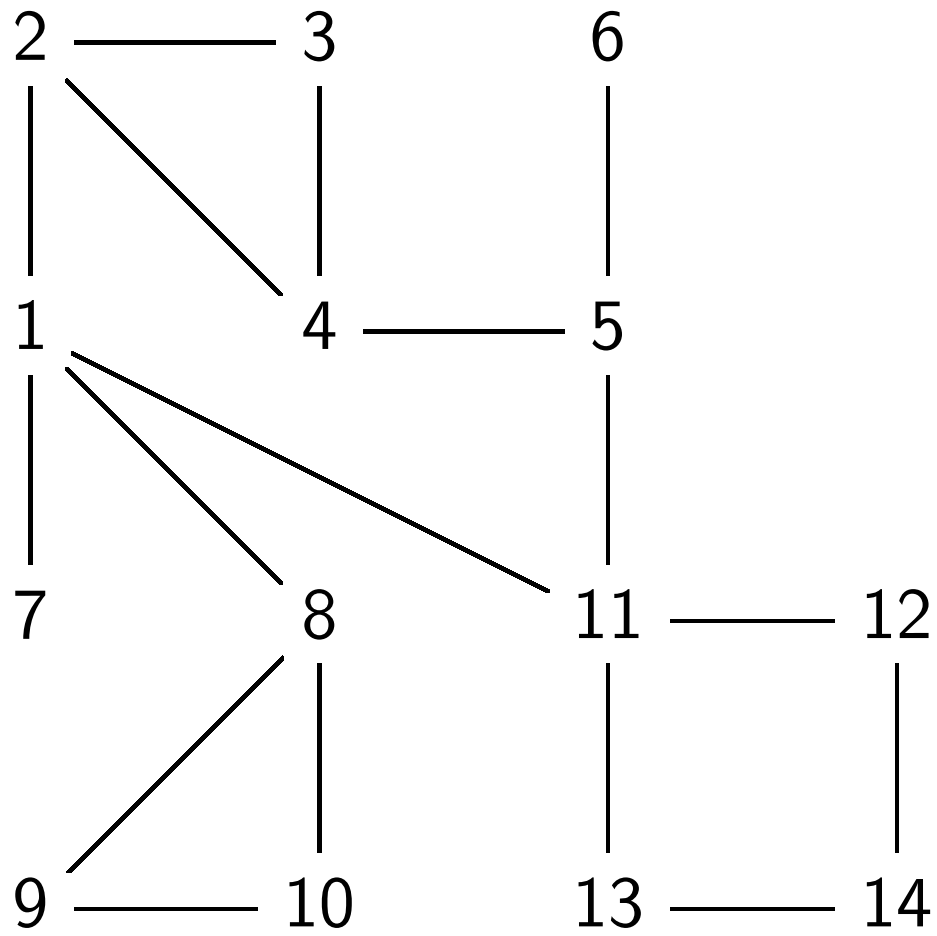

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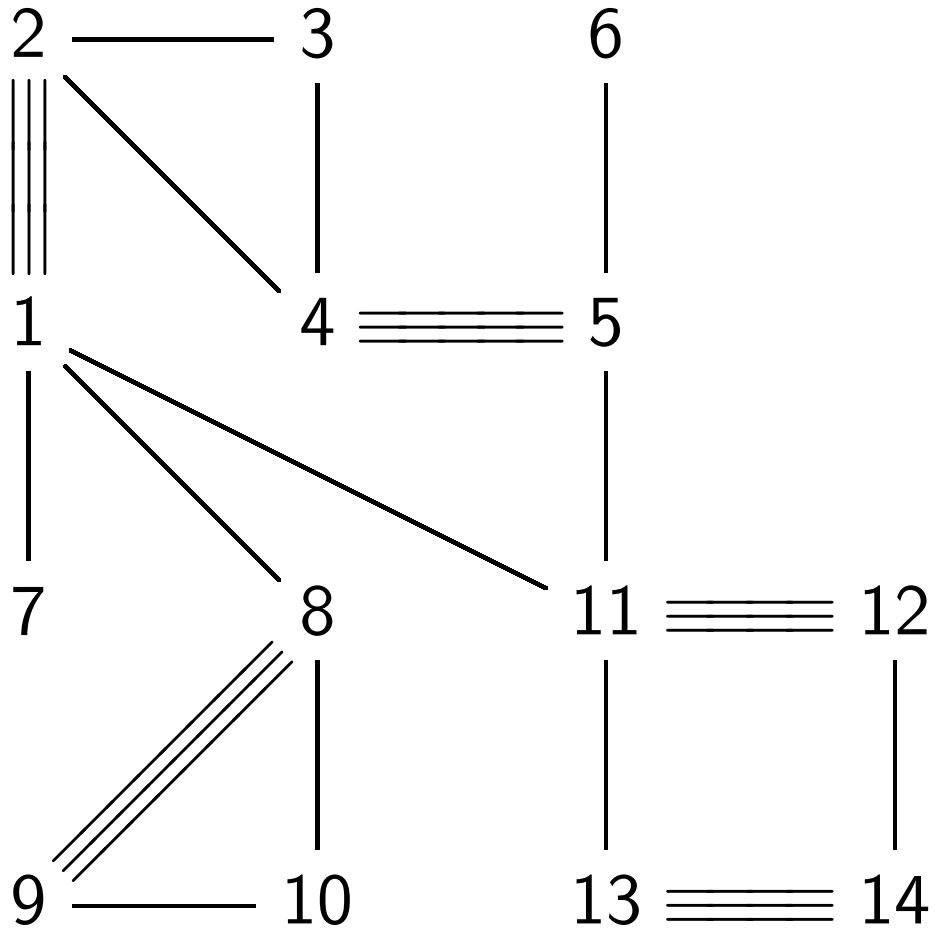

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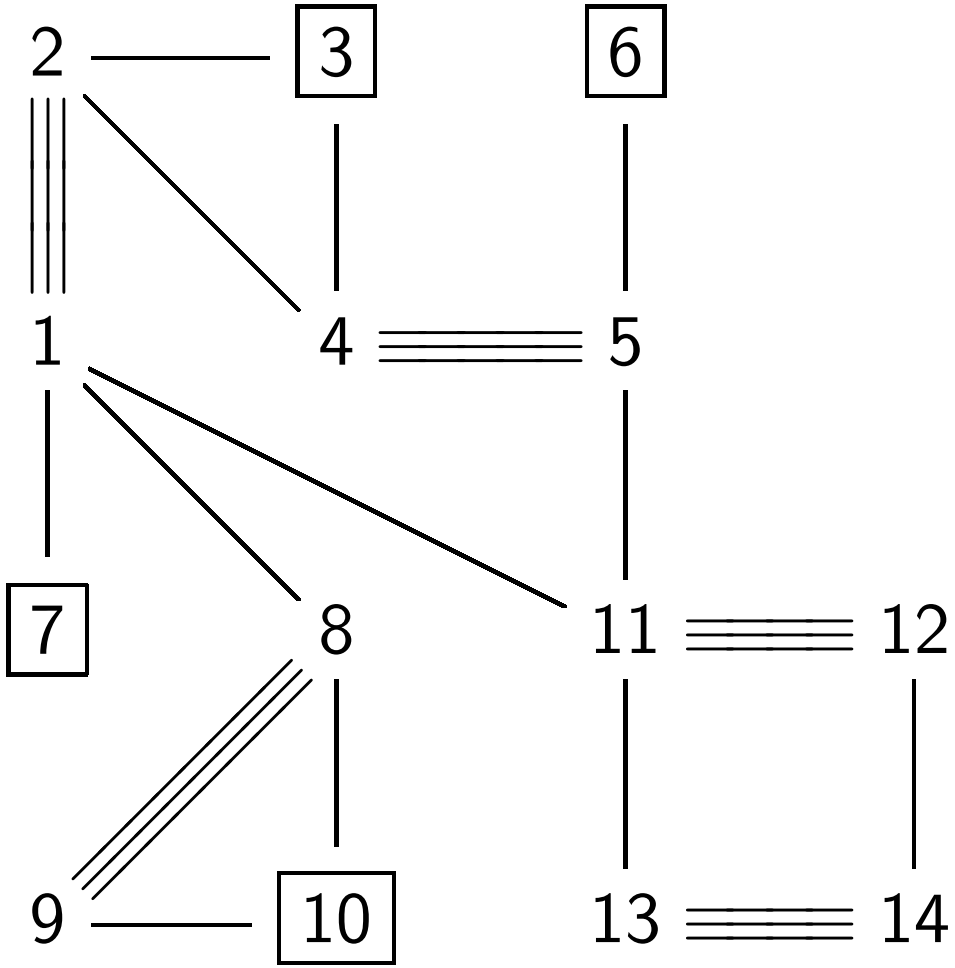

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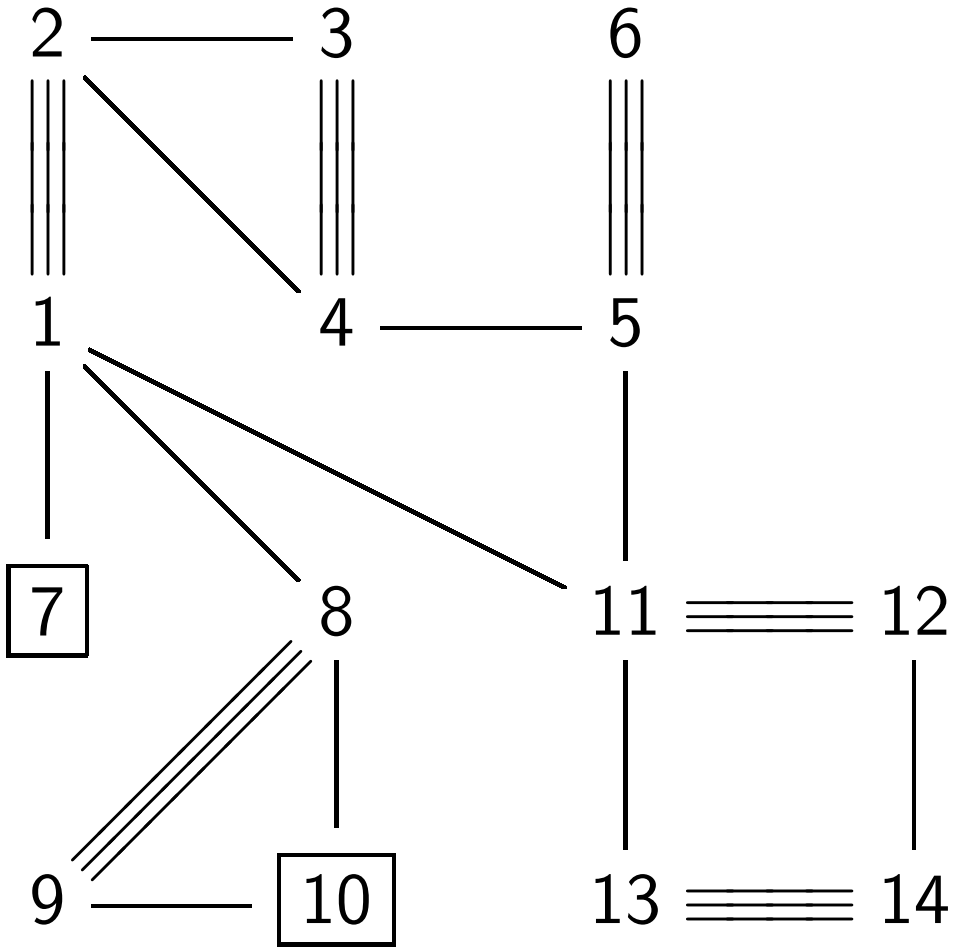

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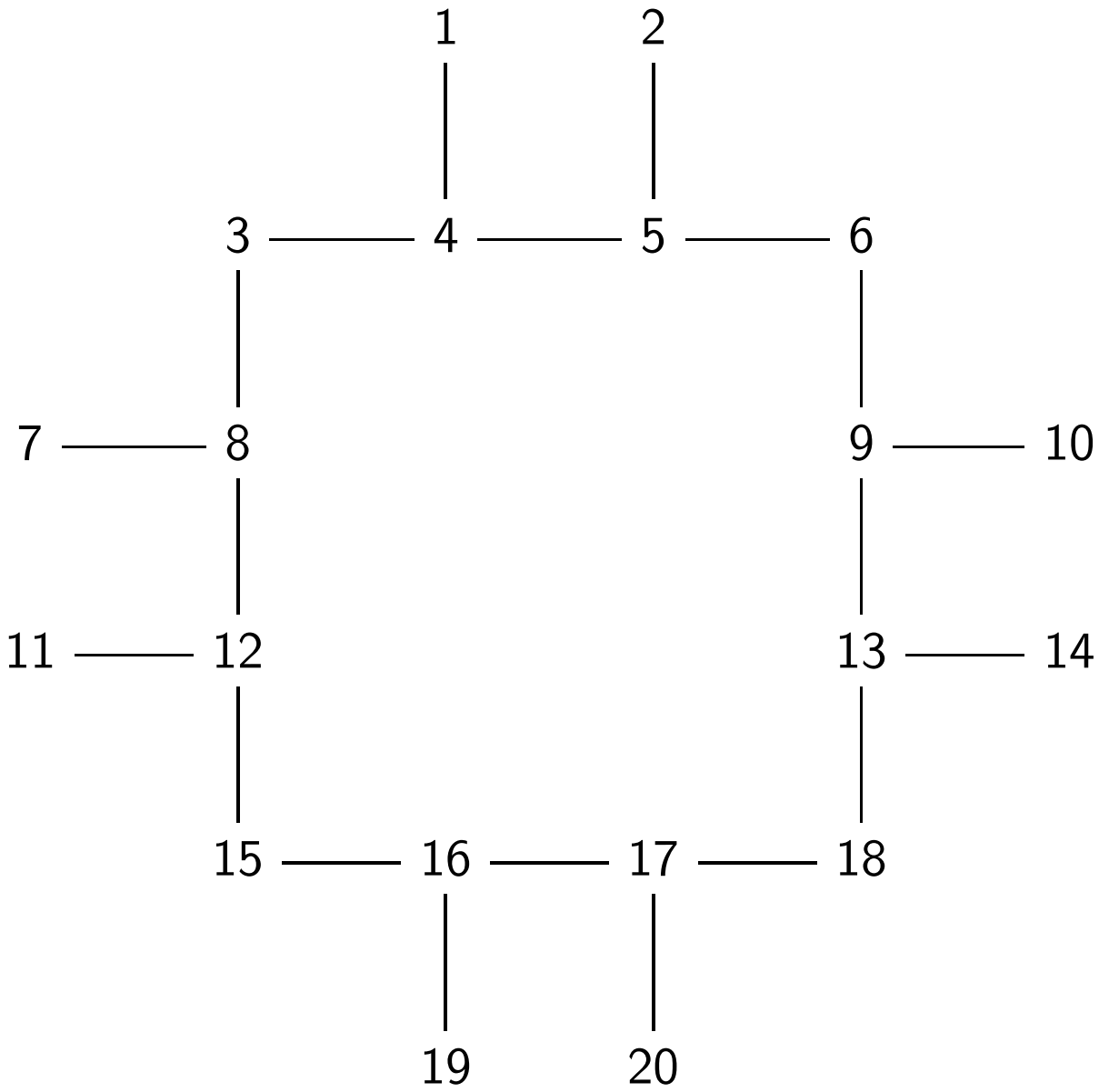

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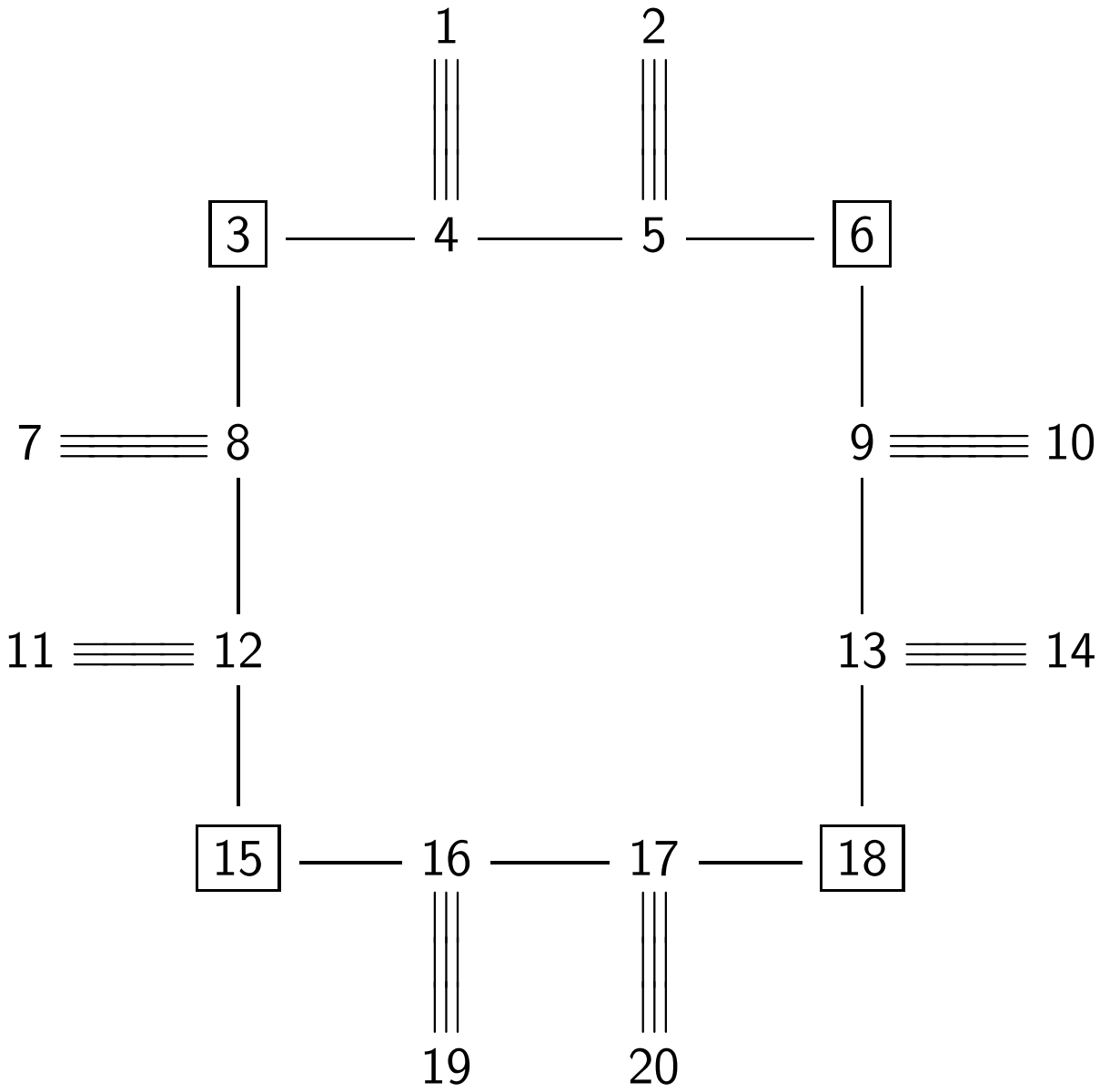
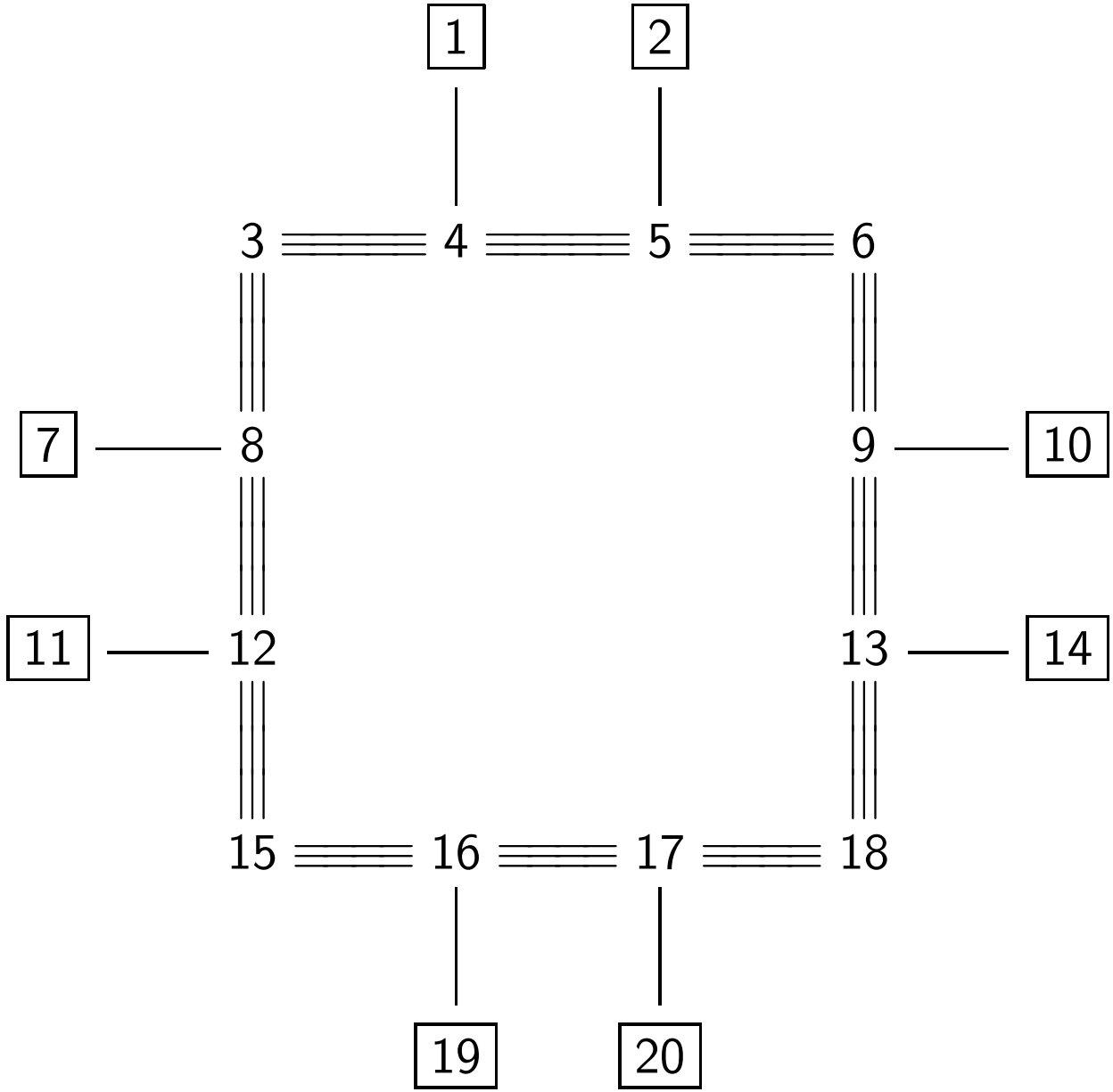


Counter-example with more than two-way exchange



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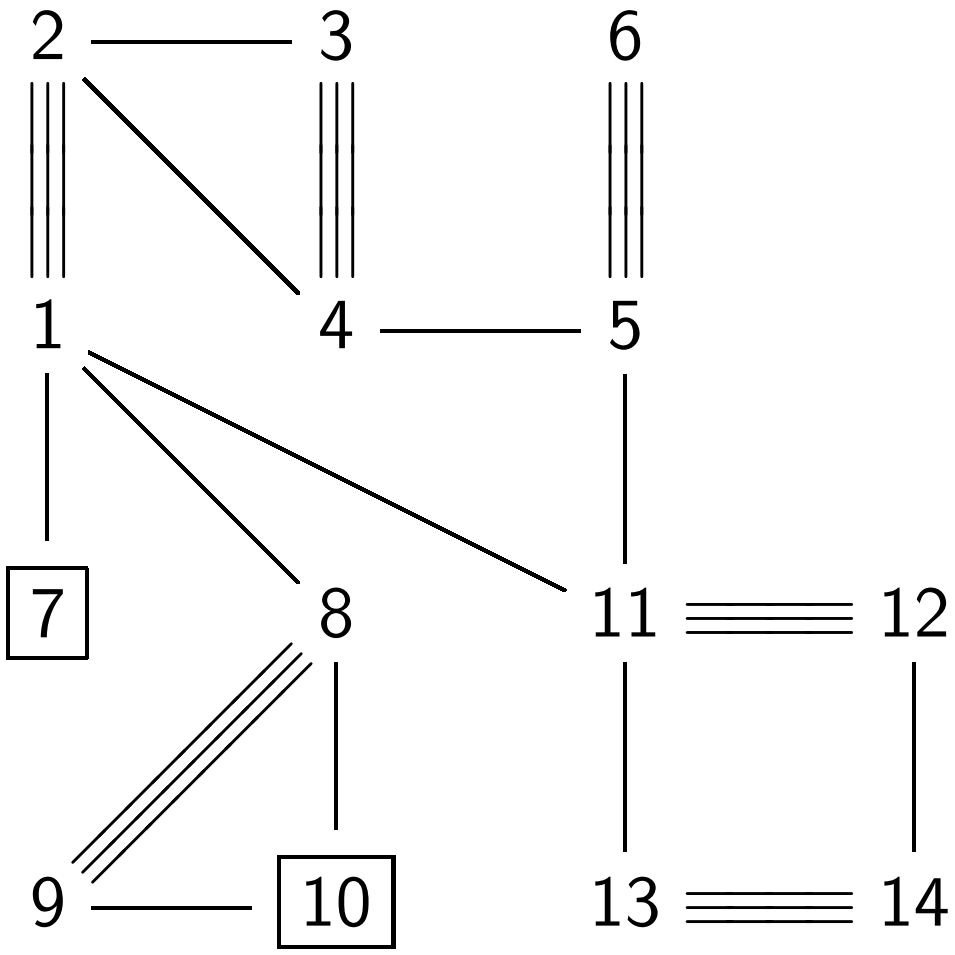

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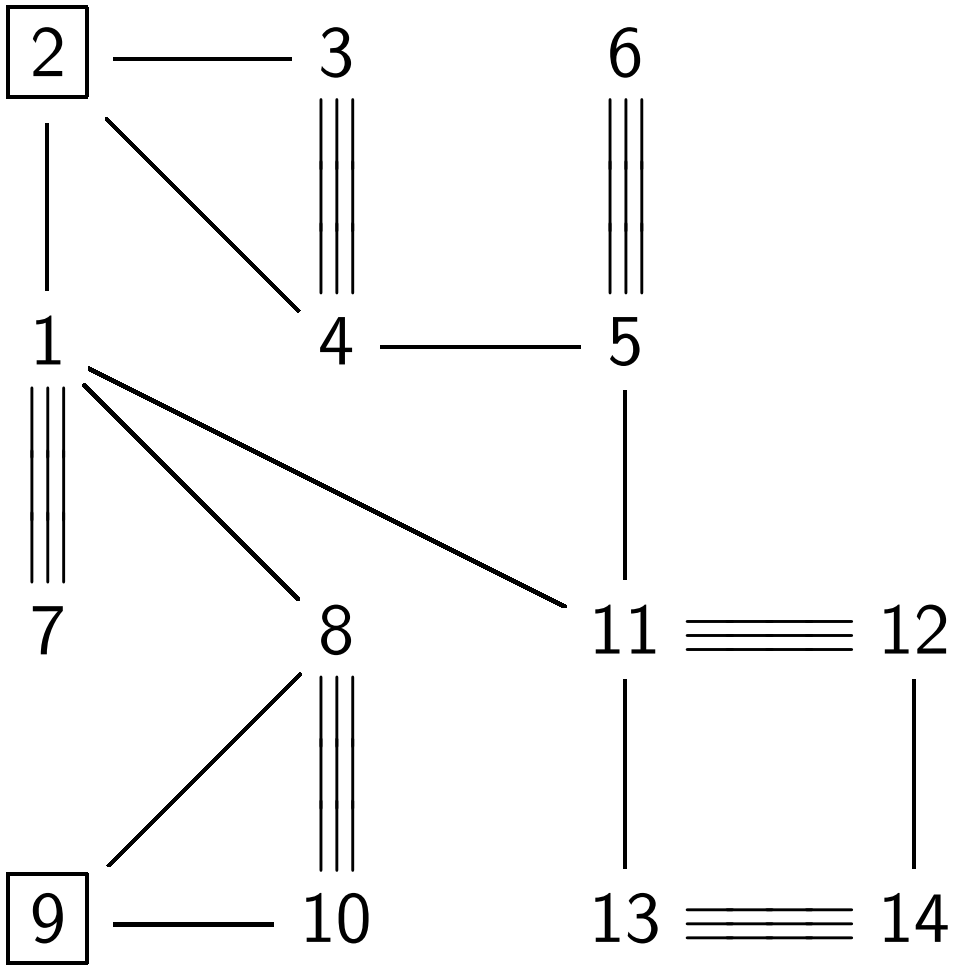

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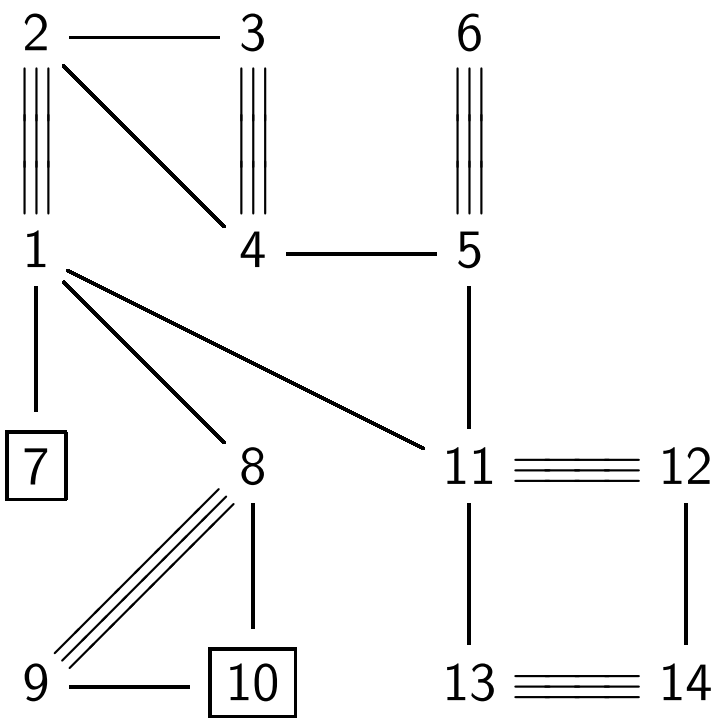

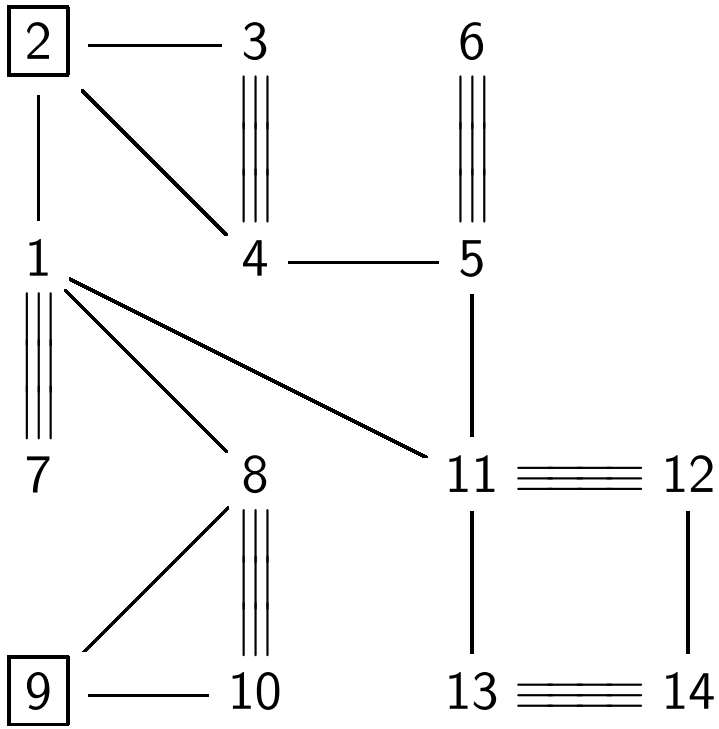

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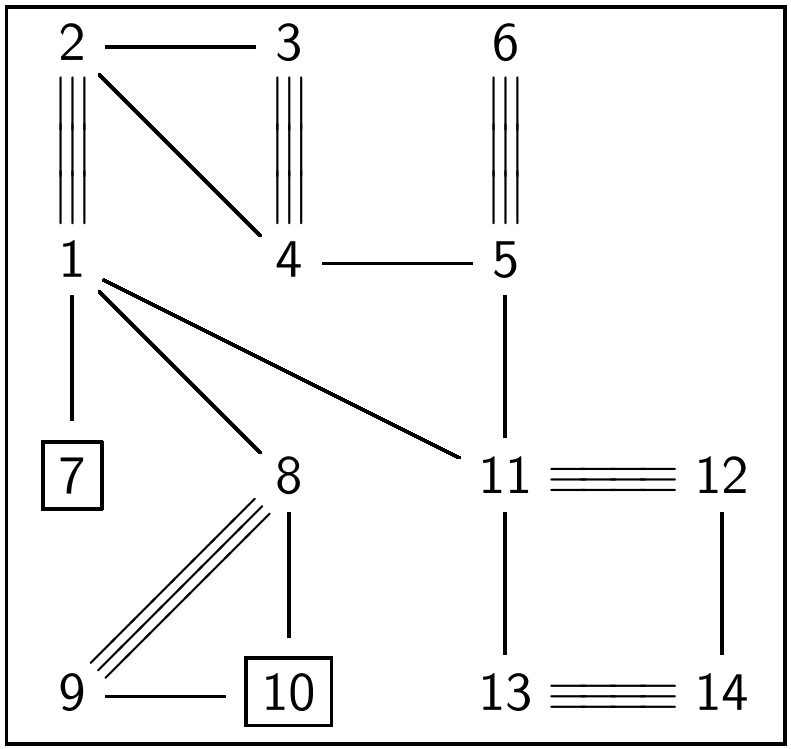

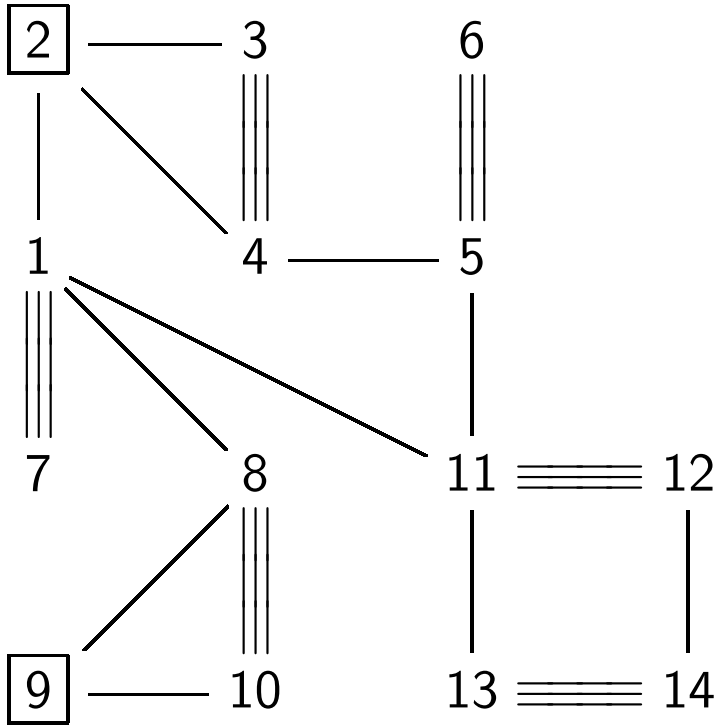

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- The one-to-one case : so-called "marriage problem". (pioneers : Gale, Shapley, Knuth)

Two mechanisms

Mechanism (DA **Women** proposing)

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\begin{array}{llll}
W_1: & m_3 & m_1 & m_2 \\
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  - Reasonable ? "Responsive" : preferences over subsets generated from pref over students.
  - If school s_1 prefers students t_1 to t_2, then also prefer subset $T \backslash \{t_1\} \cup \{t_2\}$ to subset T.

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